Math 113 Exam 1 Practice

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Exam 1 will cover sections 6.1-6.5 and 7.1-7.5 This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section give the answers of the questions in section 2.

6.1: Areas between Curves

Here are the important steps to keep in mind:

- Begin by sketching the curves together in one coordinate plane. This is the most important step; be sure you can do it without using a calculator.
- Choose which variable to integrate with respect to. If the curves give y as a function of x (e.g., $y = x^3 - x + 1$), you will want to integrate with respect to x, while if the curves give x as a function of y (e.g., $x = e^y + y$), you will want integrate with respect to y. In some cases it may be possible to solve for either variable (e.g., $2x + 3y = 5$).
- Find the intersection points of the curves. These will determine the bounds of the integral(s). If you are integrating with respect to x, then you are interested in the x coordinates of the intersection points, while if you are integrating with respect to y, then you are interested in the y coordinates of the intersection points.
- Keep in mind that the area of the region is found by splitting it up into thin approximately rectangular pieces; the width of these rectangles become the dx or dy in the integral, while the height (or length) of the rectangles are found by subtracting the y-coordinate of the curve on top from the y-coordinate of the curve on bottom (or, if you are integrating with respect to y, by subtracting the x-coordinate of the curve on the right from the x-coordinate of the curve on the left).
- It may be necessary to split the region into two or more pieces. But this can sometimes be avoided by integrating with respect to the other variable.

6.2 & 6.3: Volumes

The guidelines above also apply to these problems. For solids of revolution, you will have to decide whether to use the method of slicing or cylindrical shells:

- If the axis along which you are integrating is parallel to the axis of rotation, use the method of slicing. If the axis along which you are integrating is perpendicular to the axis of rotation, use the method of cylindrical shells.

- Keep in mind that the volume of the solid is found by splitting up the region into thin approximately rectangular pieces and considering the volume swept out by each of these rectangles when they are revolved about the appropriate axis. Each small rectangle (which in the limit becomes simply a line segment) will sweep out either a washer or a cylindrical shell. Visualizing this process will make it easy to determine which method is applicable for a given problem.

- The area of a washer is $\pi(r_{out}^2 - r_{in}^2)$, where r_{out} is the radius of the outer circle and r_{in} is the radius of the inner circle; r_{out} and r_{in} are found by measuring the distance from the two curves to the axis of rotation.

- The area of a cylindrical shell is $2\pi rh$, where r is the radius (distance to the axis of rotation) and h is the height of the cylinder (found by subtracting: the top curve minus the bottom curve, or the right curve minus the left curve, depending on which variable we are integrating with respect to).

- Some solids do not result from revolving a region about an axis. For such solids, the method of cylindrical shells is not possible. The volume must be found by slicing. To do this, choose an axis along which to integrate; identify the type of cross sections perpendicular to this axis (whether they be triangles, squares, disks, etc.); and calculate the area of such cross sections. The volume is the integral of these areas.

6.4: Work

The work done by a force on an object is given by $W = Fd$, where F is the magnitude of the force and d is the displacement of the object in the direction of the force. In each problem, it is important to first **establish a coordinate system**: decide where to place the origin and in which direction to point the positive axis.

- If a varying force $F(x)$ acts on an object, then the work is calculated by integrating the force over the distance travelled:
$$W = \int_a^b F(x)\,dx.$$
Remember, we found this formula by dividing the axis of the object's movement into small intervals and calculate the work done by the force on the object as it moves over the length of each small interval.

- If the object consists of parts, each of which is to be displaced by a different amount, divide the object into small parts and calculate the work done by the force in moving each part of the object to its final location. For example, in the case of the movement of a liquid out of a tank, we take the cross sectional area of the tank at a specific height (or depth) and the work done at that height is

  Weight density times cross sectional area times Δh times distance travelled.

  If $D(h)$ represents the height travelled at depth h, and $A(h)$ is the cross section area, the work becomes
$$W = \int_a^b \omega D(h)A(h)\,dh$$
where ω is the weight density.

6.5: Average Value of a Function

Recall that the Mean Value Theorem for integrals states that for a continuous function on a closed interval, there is a c with

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The right hand side of the above is called the **average value** of f over $[a, b]$. You are expected to do the following:

a) Find the average value of the following functions over the specified interval

b) Find a c in the interval on which the function achieves its average value.

7.1: Integration by Parts

$$\int u\;dv = uv - \int v\;du$$

- Integration by parts is most often useful when integrating a function of the form $x^n e^x$, $x^n \sin x$, $x^n \cos x$, $x^n \ln x$. If possible, you want to choose u to be a function that becomes simpler when differentiated, and dv to be a function that can be readily integrated. This usually means you should choose $u = x^n$. (But in the case $x^n \ln x$, choose $u = \ln x$).

- Integration by parts is also useful for integrating inverse functions such as $\sin^{-1} x$, $\tan^{-1} x$, $\ln x$ or functions involving these as factors. In this case, you should choose $u = \sin^{-1} x$, $u = \tan^{-1} x$, or $u = \ln x$ accordingly, even if there are no other factors in the integrand (i.e., you can set $dv = dx$).

7.2: Trigonometric Integrals

- For $\int \sin^m x \cos^n x\;dx$:
  If n is odd, save one $\cos x$ and convert the rest to sin using $\cos^2 x = 1 - \sin^2 x$
  If m is odd, save one $\sin x$ and convert the rest to cos using $\sin^2 x = 1 - \cos^2 x$
  If both m and n are even, use the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

- For $\int \tan^m x \sec^n x\;dx$:
  If n is even, save one $\sec^2 x$ and convert the rest to tan using $\sec^2 x = \tan^2 x + 1$
  If m is odd, save a $\sec x \tan x$, and convert the rest to sec using $\tan^2 x = \sec^2 x - 1$.
  If m is even and n is odd, convert everything to sec and integrate by parts with $dv = \sec^2 x$.
  (This last case will require "solving" for the desired integral.)

- For $\int \tan^n x\;dx$, convert one $\tan^2 x$ to $\sec^2 x - 1$ and split the problem into two integrals.

- $\int \sec x\;dx = \ln|\sec x + \tan x| + C$.

- A similar strategy applies for $\int \cot^m x \csc^n x\;dx$.

7.3: Trigonometric Substitution

- If the integrand involves $\sqrt{a^2 - b^2x^2}$, use $x = \frac{a}{b}\sin\theta$.
- If the integrand involves $\sqrt{b^2x^2 + a^2}$, use $x = \frac{a}{b}\tan\theta$.
- If the integrand involves $\sqrt{b^2x^2 - a^2}$, use $x = \frac{a}{b}\sec\theta$.
- If the integrand involves $\sqrt{ax^2 + bx + c}$, complete the square to get it into the form $\sqrt{a(x-h)^2 + k}$. After factoring out the a and applying the substitution $u = x - h$, the integrand will then fit one of the three forms above.
- Avoid using a trigonometric substitution when a regular u-substitution is possible.

7.4: Integration of Rational Functions by Partial Fractions

Rational functions consist of fractions of polynomials. We can split rational functions into simpler pieces by partial fractions. Remember that partial fraction decompositions are based on linear and quadratic factors in the denominator. For each linear factor, we have a term with a constant in the numerator and the factor in the denominator. For each irreducible quadratic, we have a term with a linear function in the numerator and the quadratic in the denominator. For example,

$$\frac{x^2 - x + 2}{(x-1)(x+2)(x^2+x+3)} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{Cx+D}{x^2+x+3}. \tag{1}$$

We just need to determine the values of A, B, C and D. This is done by plugging in values for x: You need to plug in as many numbers as you have constants. Using some numbers, (like -2 and 1 in this case) makes your life easier, but any four numbers will do. Notice that if we multiply equation 1 by the denominator on the left side, we get

$$x^2 - x + 2 = A(x+2)(x^2+x+3) + B(x-1)(x^2+x+3) + (Cx+D)(x-1)(x+2). \tag{2}$$

Letting $x = -2$ in equation 2 gives $8 = -15B$, so $B = -8/15$. Letting $x = 1$ gives $2 = 15A$, so $A = 2/15$. Letting $x = 0$ gives $2 = 5A - 3B - 2D$. Knowing A and B helps us to find D. Finally, if $x = -1$, $4 = 9A - 6B + 2C - 2D$, allowing us to solve for C.

Remember that repeated factors must give repeated terms with increasing exponent in the denominator. For example,

$$\frac{x^3 + 2x^2 + 2x - 5}{(x-2)^3(x^2+9)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Finally, remember partial fractions only works if the degree in the numerator is less than the degree in the denominator. Otherwise, you need to divide and use partial fractions on the remainder.

7.5 Strategy for Integration

You may have noticed that we have really only used two techniques of integration in chapter 7: substitution and integration by parts. Everything else is just manipulation of those two techniques. Luckily, most integrals that require a specific technique have patterns that help remind us. Those patterns have been discussed earlier. You are expected to know them.

A general strategy for attacking unknown integrals is as follows:

1. Can substitution be used to simplify the integral? If so, do this first. (Note that in this section, substitution is often just the first step. After the substitution, some other technique needs to be applied.)

2. If you cannot use substitution, or you have used it but the integral is still not simple enough, look for patterns.
   (a) If the integral is a product of trig functions, use the patterns we learned in 7.2 to tackle it.
   (b) If the integral has a sum or difference of squares in the integrand, use trig substitution.
   (c) If the integrand is a rational function, use partial fractions.
3. If you cannot see one of the above patterns, and cannot use substitution, try integration by parts.

Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

For questions 1 to 5, find the area of the region enclosed by the given curves.

1. $y = x^3 - x + 1$, $y = 1$, $2 \leq x \leq 3$
2. $y = x^2 - 1$, $y = 2 - x - x^2$
3. $x^2 + (y-1)^2 = 1$, $y = x$
4. $y = x^2$, $y = 3x^2$, $2x + y = 1$, $x \geq 0$
5. $y = \sqrt{x^2+1}$, $x = 0$, $x = 1$, $y = -1$

   In problems 6 to 10, find the volume of the solid obtained by revolving the region bounded by the given curves about the given axis.

6. $y = 4 - x^2, y = 0$; about the x-axis
7. $x = -y - 1, x = e^y, 0 \leq y \leq 1$; about the x-axis
8. $y = \sin x + 1, y = -1, x = 0, x = \frac{\pi}{4}$; about the line $x = -1$
9. $y = \sec x, y = 0, x = 0, x = \frac{\pi}{3}$; about the line $y = 3$
10. $x = \sqrt{y} + y, x = 0, y = 0, y = 1$; about the line $y = -1$

    In problems 11 to 12, find the volume of the described solid.

11. The base of the solid is the unit disk $x^2 + y^2 \leq 1$. Cross sections perpendicular to the x-axis are squares.
12. The base of the solid is the region enclosed by the curves $x + y = 1$, $x + y = -1$, $x - y = 1$, and $x - y = -1$. Cross sections perpendicular to the x-axis are equilateral triangles.
13. A chain weighing 3 lb/ft is used to lift a 500 lb object a height of 20 ft, to the level of the top of the chain. Find the work done.
14. A particle is moved along the x-axis from the origin a distance of 5 meters by a force which varies depending on the position of the particle. When the particle is at x, the force is $(2x+1)/(x+1)$ newtons in the positive direction. Find the work done by the force on the particle. [Hint: Use long division to simplify the integrand.]
15. A force of 20N is required to hold a spring stretched 40cm, while a force of 30N is required to hold it stretched 45cm. How much work is required to stretch the spring from 50cm to 60cm? [Hint: Find the natural length of the spring.]
16. The great pyramid of Giza consists of approximately 2 million stones, each weighing 1.5 tons (3000 lb). The pyramid is 450 ft high with a square base measuring 750 ft. Find the work required to lift the stones into place from ground level. [Hint: To simplify your calculations, work the prob-

lem symbolically; only plug in the given numbers at the last step.]

For the questions 17 to 19, find the average value of the function over the interval, and find the value c where $f(c)$ is equal to the average value (or show why no such value exists).

17. $f(x) = x^2 - x + 2, [-1, 2]$

18. $f(x) = \sin^2 x, [0, \frac{\pi}{2}]$

19. $f(x) = 1/(x+1), [0, 2]$

For problems 20 to 46, evaluate the integral.

20. $\int x \cos x \; dx$

21. $\int_0^{\frac{\pi}{2}} x \sin x \; dx$

22. $\int_0^1 x^2 e^x \; dx$

23. $\int_0^1 \sin^{-1} x \; dx$

24. $\int 2x \tan^{-1} x \; dx$

25. $\int \frac{\ln x}{x^2} \; dx$

26. $\int e^x \cos x$

27. $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} x^3 \sin(x^2) \; dx$ [Hint: First use a u-substitution]

28. $\int_0^{\frac{\pi}{6}} \sin^2 x \cos^3 x \; dx$

29. $\int_0^{\frac{\pi}{3}} \cos^4 x \; dx$

30. $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \; dx$

31. $\int \tan^2 x \sec^2 x \; dx$

32. $\int \tan^6 x \; dx$

33. $\int \tan^2 x \sec x \; dx$

34. $\int \frac{1}{x\sqrt{x^2+1}} \; dx$

35. $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} \; dx$

36. $\int \frac{1}{x^2\sqrt{25+x^2}} \; dx$

37. $\int \frac{x}{\sqrt{x^2+2x}} \; dx$

38. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{x^2+x+\frac{5}{4}}} \; dx$

39. $\int \frac{x}{\sqrt{4x-x^2}} \; dx$

40. $\int_0^1 \frac{2x-1}{x^2-x-2} \; dx$

41. $\int \frac{6}{x^3+2x^2+x} \; dx$

42. $\int \frac{x^2+x-5}{x^2-1} \; dx$

43. $\int \frac{x-1}{x^3+x} \; dx$

44. $\int \frac{x^3}{x^4+2x^2+1} \; dx$

45. $\int_9^{16} \frac{\sqrt{x}}{x-4} \; dx$ [Hint: Use a rationalizing substitution]

46. $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x} \; dx$ [Hint: Use the substitution $t = \tan(\frac{x}{2})$]

Answers

1. $\frac{55}{4}$
2. $\frac{125}{24}$
3. $\frac{\pi}{4} - \frac{1}{2}$
4. $\frac{4}{3}\sqrt{2} - \frac{50}{27}$
5. $1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1)$
6. $\frac{512}{15}\pi$
7. $\frac{11}{3}\pi$
8. $\pi(\frac{\pi^2}{8} - \frac{\sqrt{2}}{4}\pi + \pi + 2)$
9. $\pi(6\ln(2+\sqrt{3}) - \sqrt{3})$
10. $\frac{19}{5}\pi$
11. 4
12. $\frac{2}{3}\sqrt{3}$
13. 10600 ft-ln
14. $10 - \ln 6$
15. 5 J
16. 675,000,000 ft-lb
17. $f_{ave} = \frac{5}{2}, c = \frac{1\pm\sqrt{3}}{2}$
18. $f_{ave} = \frac{1}{2}, c = \frac{\pi}{4}$
19. $f_{ave} = \frac{1}{2}\ln 3, c = \frac{2}{\ln 3} - 1$
20. $x\sin x + \cos x + C$
21. 1
22. $e - 2$
23. $\frac{\pi}{2} - 1$
24. $x^2\tan^{-1}x - x + \tan^{-1}x + C$
25. $-\frac{1}{x}\ln x - \frac{1}{x}$
26. . $\frac{1}{2}e^x(\sin x + \cos x) + C$
27. $\frac{1}{2}(\pi - 1)$
28. $\frac{17}{480}$
29. $\frac{\pi}{8} + \frac{7}{64}\sqrt{3}$
30. $\frac{1}{2}$
31. $\frac{\tan^3 x}{3} + C$
32. $\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + \tan x - x + C$
33. $\frac{1}{2}(\sec x\tan x - \ln|\sec x + \tan x|) + C$
34. $\ln|x| - \ln|\sqrt{x^2+1} + 1| + C$
35. $\frac{40}{3}$
36. $\frac{-\sqrt{x^2+25}}{25x} + C$
37. $\sqrt{x^2+2x} - \ln|x + 1 + \sqrt{x^2+2x}| + C$
38. $\sqrt{2} - 1 - \frac{1}{2}\ln(\sqrt{2}+1)$
39. $2\sin^{-1}(\frac{x}{2} - 1) - \sqrt{4x - x^2} + C$
40. 0
41. $6\ln|x| - 6\ln|x+1| + \frac{6}{x+1} + C$
42. $x - \frac{3}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + C$
43. $-\ln|x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + C$
44. $\frac{1}{2}(\ln|x^2+1| + \frac{1}{x^2+1}) + C$
45. $2(\ln 5 - \ln 3 + 1)$
46. $\frac{2\sqrt{3}}{9}\pi$